A FRACTAL MODEL OF OCEAN SURFACE SUPERDIFFUSION

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The transport of surface pollutants in the coastal zone is modelled using a modified particle tracking diffusion model. The new model uses fractional Brownian motion (fBm) functions to produce superdiffusive spreading of the synthesised pollutant clouds. The model is tested on a numerical model of a coastal bay recirculation zone.

1 Introduction

The spread of pollutants in the environment is a topic of much research. Many environmental pollutants are transported via turbulent fluids (in rivers, the oceans and the atmosphere). The overall spreading of the pollutants in these turbulent flowfields is generally non-Fickian (in fact superdiffusive) over some, if not all, length scales under consideration. Traditional methods of modelling diffusive processes in the environment rely on either solving an advection-diffusion equation on a computational grid or using a particle-tracking technique[1-3]. Both methods lead to solutions which are Fickian in nature. The spatial correlations which exist over significantly large scales result in a Lagrangian memory effects within the flow field. Recently the authors have developed a particle-tracking technique that produces non-Fickian diffusion[4-6]. The technique works by incorporating fractional Brownian motion (fBm) trajectories within the particle tracking model. The work is motivated by the recent discovery that ocean surface drifter trajectories are fractal in nature[7-9]. In this paper previous work is expanded and the simulation of coastal diffusion in a numerically generated bay flow model is conducted.

2 The Non-Fickian Particle Tracking Model

Particle tracking models work by releasing a large number of massless marked particles into a known flow-field. These are then diffused due to a Fickian diffusion model and advected according to the spatial distribution of the flow-field. Each particle represents a portion of the mass of the contaminant, and the ensemble particle cloud can be converted into a spatial concentration distribution. The authors generate non-Fickian diffusive behaviour of particle clouds within a particle tracking diffusion model by specifying each particle path as a fractional Brownian motion (fBm). A generalisation of Brownian motion, fBm is a random fractal function defined by Mandelbrot and Van Ness[10] which exhibits long term correlation over all scales. In this section we briefly give the algorithm for the generation of fBm, more specific details of the model are given by the authors elsewhere, see for example[11-13]. Fractional Brownian motion is defined as

$$B_H(t) = \frac{1}{\Gamma(H+\frac{1}{2})}\left[\int_{-\infty}^{0}\left[(t-t')^{H-\frac{1}{2}} - (-t')^{H-\frac{1}{2}}\right].R(t').dt' + \int_0^t.(t-t')^{H-\frac{1}{2}}.R(t').dt'\right], \quad (1)$$

where Γ is the gamma function and $R(t)$ is a continuous white noise function. A discrete approximation to Eq. (1) may be generated in two steps: Firstly, incremental steps in the fBm walk are calculated using

$$B_H(t_i) - B_H(t_{i-1}) = \frac{1}{\Gamma(H+\frac{1}{2})}\left[\left[\sum_{j=i-M}^{i-2}\left((i-j)^{H-\frac{1}{2}} - (i-j-1)^{H-\frac{1}{2}}\right).R(t_j)\right] + R(t_{i-1})\right], \quad (2a)$$

where M is a finite memory; and $R(t_j)$ are discrete random numbers with a Gaussian distribution of known standard deviation and zero mean. (Note that simpler distributions may be used.) These incremental steps are known as fractional Gaussian noise (fGn) - generalisations of white noise. The fGn is then summed to generate fBm at discrete times $t_i \, (=\Delta t \times i)$

$$B_H(t_i) = \sum_{k=1}^{i}\left[B_H(t_k) - B_H(t_{k-1})\right]. \quad (2b)$$

Eqs. (2a) and (2b) may then be used to specify the displacement-time behaviour of individual particles in each spatial dimension of the diffusive problem under investigation. This requires an independent realisation of the fBm for each spatial co-ordinate. Carrying this out for each particle in a diffusing cloud results in scaling of standard deviation of the cloud, σ_c, of the form

$$\sigma_c = \left(2.K_f.t\right)^H \quad (3)$$

(here we do not include advection), where K_f is the fractal (anomalous) diffusion coefficient and H is the Hurst exponent[14].

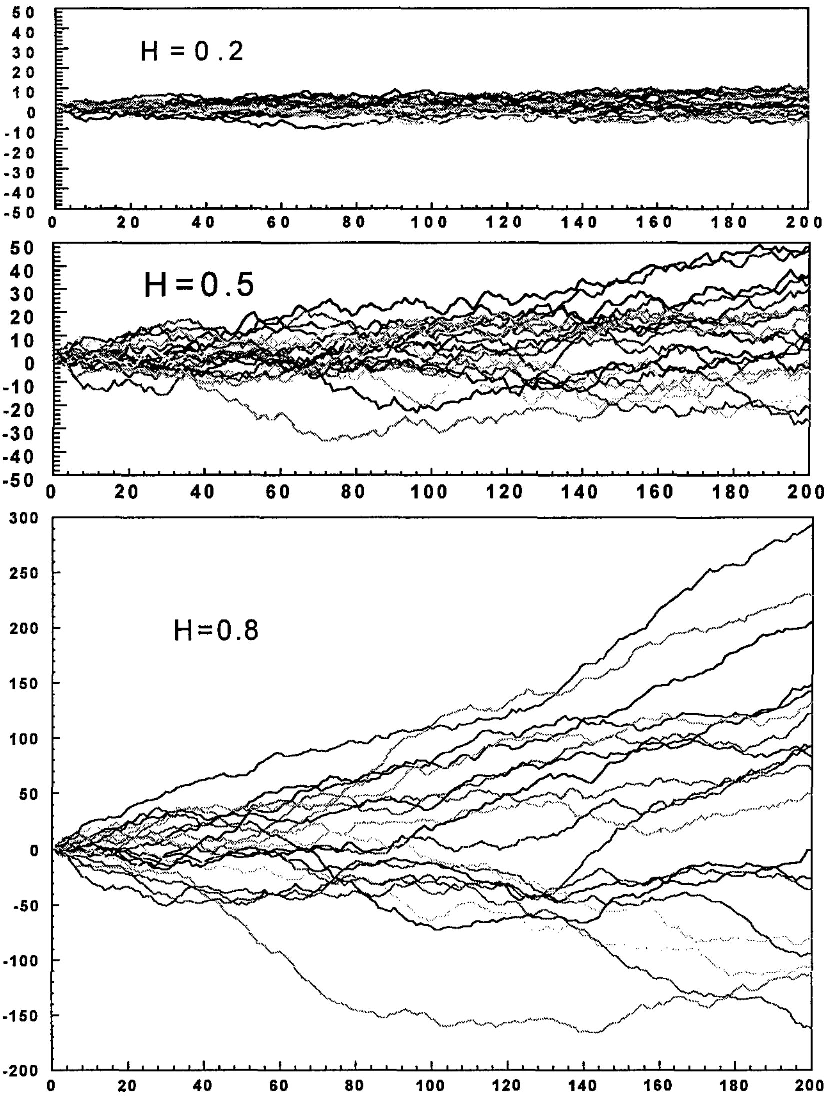

Figure 1: Clouds of 20 fBm Traces H = 0.2, 0.5, 0.8. (Note: Different Scales on Vertical Axes.).

Fig. 1 shows the diffusion of clouds of fBm (H=0.2, 0.5 and 0.8) from a point source. The respective subdiffusive and superdiffusive nature of the H=0.2 and 0.8 fBm is evident in the plot. In order to generate the fBm in two (or more) spatial directions an fBm trace is used for each independent co-ordinate. Figure 2 contains 2D fBm spatial trajectories for single particles. Again the subdiffusive nature of the H=0.2 path and superdiffusive nature of the H=0.8 path can be seen. The particle tracking model works by releasing many 1000's of particles utilising their ensemble spreading behaviour.

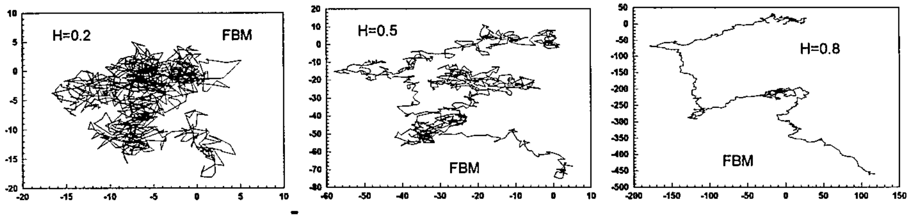

Figure 2: Two-Dimensional fBm trajectories.

The finite memory, M, of the synthesised fBm is an important factor that must be taken account of when generating the fBm using the above method. Memories that are too low result in a poor representation of the fBm over the problem timescale. Figure 3 shows the effect of low memory on the resulting power law of the diffusing cloud. In the figure a 100 step fBm is generated using memories from 100 to 2000 steps, i.e. from one times the length of each component trace to twenty times the length of each trace. The gradual straightening of the curve is evident from the plot indicating that the synthesised fBm realises the scaling power law (Eq. 3) as the memory increases. The authors have found that a memory of approximately 5 times the problem lengthscale is suitable for most practical applications.

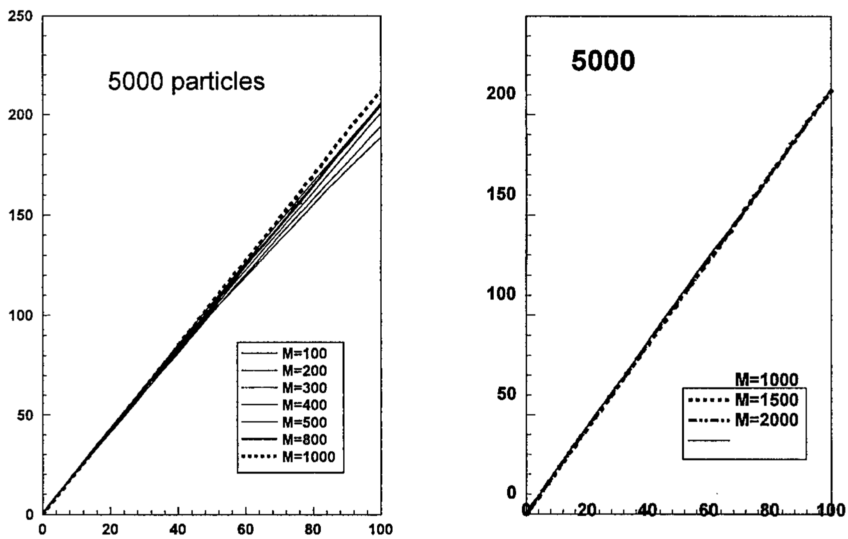

Figure 3: $\sigma^{\frac{1}{H}}$ versus time for various values of memory M.

3 Dispersion in a Coastal Bay

Fig. 4 shows a numerically generated surface velocity field for a coastal bay with a North-South flow. This main flow causes a recirculation zone within the bay itself. In such complex flow-fields the relative change in the velocity vectors (magnitude and direction) can cause additional dispersion of the pollutant cloud to take place. This effect is known as shear dispersion.

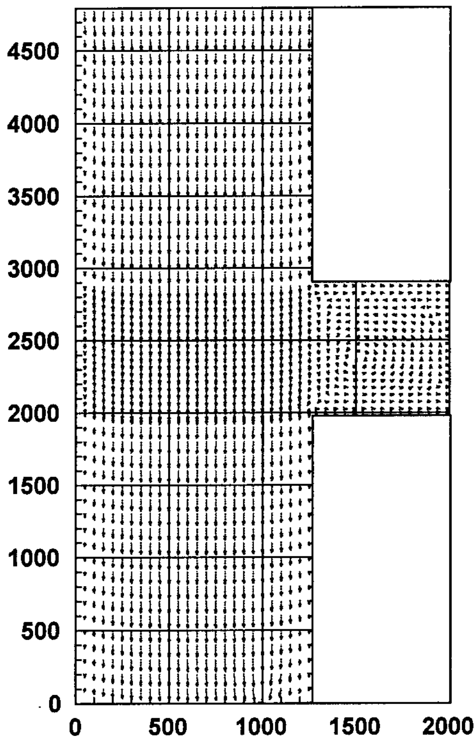

Figure 4: Coastal Bay Model, Surface Velocity Vector Plot.

Fig. 5 shows the comparison of two synthesised pollutant clouds, released in the bay, as they are dispersed over time. The left hand plots for each pair contains the isoconcentration contours of a Fickian cloud (H=0.5) and the right hand plots for a superdiffusive (H=0.8) cloud. Within the bay itself we see that the non-Fickian cloud spreads much more rapidly than the (traditional) Fickian cloud. This rapid spreading can result in areas being affected that would otherwise escape contamination with the Fickian model. In addition, lower concentrations are reached much sooner with the superdiffusive cloud but a larger area is affected. This has implications for the modelled biological impact of the pollutant on the environment, especially for those organisms whose mortality depends on a threshold concentration of contaminant.

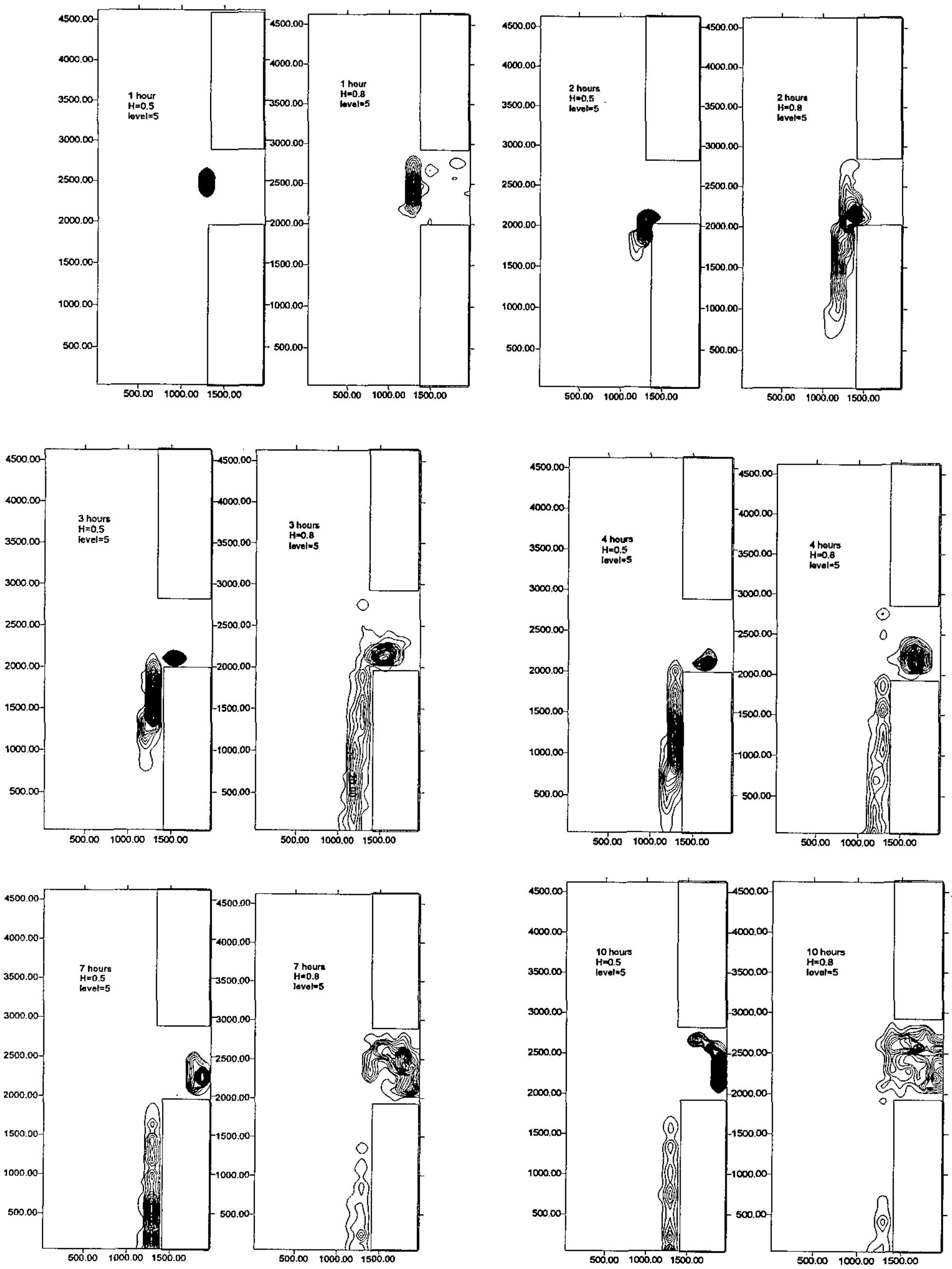

Figure 5: Comparison of Fickian (H = 0.5) and Non-Fickian (H = 0.8) Contaminant Clouds Released from (1300, 2800) at Time Zero. Plots show location of cloud after 1, 2, 3, 4, 7, 10 hours.

4 Concluding Remarks

The non-Fickian particle tracking diffusion model developed by the authors allows for a more flexible approach than currently available to the modelling of contaminant transport within turbulent flowfields. The model has been illustrated using a coastal bay flow-field, however, it has general applications to non-Fickian diffusive processes in one, two or three dimensions. (The authors have developed a version for subsurface diffusion through material of variable hydraulic conductivities[12,15]). In the study, detailed above, the effects of shear dispersion, wind shear and tidal motions were neglected. In a fully working model these important dispersive processes would have to be taken account of.

One drawback of the model is that H is restricted to vary between 0 and 1. However, it is known that diffusive processes on the open ocean (i.e. far from boundaries) can scale with Hurst exponents up to 1.5. It is possible that by incorporating shear dispersion, super-diffusive processes may be realised with effective exponents above unity. Another drawback of the model is that the fBm generation method, given by Eqs. (2a) and (2b), requires a heavy computational effort. This is because each fGn requires a summation over M time steps, where the memory M must be of the order of (at least) five times the duration of the problem under investigation to ensure accurate non-Fickian statistics over the problem timescale. The authors have recently developed faster spectral methods of generating fBm based on the work of Yin[16]. Current work by the authors concentrates on the both the effect of shear dispersion on fBm dispersive processes and the implementation of the fBm particle tracking model to a fully three dimensional numerical model with experimental verification.

References

1. C. M. Allen, 'Numerical simulation of contaminant dispersion in estuary flows', *Proc. Roy. Soc. Lond. A*, 381, 179-192 (1992).
2. C. M. Allen, 'Particle tracking models for pollutant dispersion', in D. G. Farmer and N. J. Rycroft (eds), *Computer Modelling in the Environmental Sciences*, IMA/Clarendon, Oxford, 1991.
3. J. R. Hunter, P. D. Craig and H. E. Philips, 'On the use of random walk models with spatially variable diffusivity', *J. Comput. Phys.* 106, 366-376 (1993).
4. P. S. Addison, 'A method for modelling dispersion dynamics in coastal waters using fractional Brownian motion', *IAHR J. Hydraulic Research* 34, 549-561 (1996).
5. P. S. Addison and B. Qu, 'Non-Fickian random walk diffusion models', in'Environmental and Coastal Hydraulics: Protecting the Aquatic Habitat', *Proceedings of Theme B, Volume 1, XXVII IAHR Congress, 10-15 August, San Francisco, 1997.* ASCE, 45-50. (1997)
6. P. S. Addison, B. Qu, A. Nisbet, G. Pender, 'A non-Fickian, particle-tracking diffusion model based on fractional Brownian motion', *Int. J. Num. Meth. Fluids* 25, 1373-1384 (1997)
7. A. R. Osborne, A. D. Kirwan, A. Provenzale, and L. Bergamasco, 'Fractal Drifter Trajectories in the Kuroshio Extension', *Tellus* 41A, 416-435. (1989)
8. B. G. Sanderson, A. Goulding, and A. Okubo, 'The Fractal Dimension of Relative Lagrangian Motion', *Tellus* 42A, 550-556. (1990)
9. B. G. Sanderson and D. A. Booth, 'The Fractal Dimension of Drifter Trajectories and Estimates of Horizontal Eddy-Diffusivity', *Tellus* 43A, 334-349. (1991)

10. B. B. Mandelbrot and J. W. Van Ness, 'Fractional Brownian Motions, Fractional Noises and Applications', *SIAM Review* 10, 422-437 (1968).
11. P. S. Addison, 'Towards a non-Fickian Diffusion Model using Fractal Geometric Techniques', *ASCE J. of Hydraul. Engng*, 124(9), 877-879. (1998).
12. P. S. Addison, B. Qu, A. S. Ndumu and I. C. Pyrah, 'A particle-tracking model for non-Fickian subsurface diffusion', *Math. Geology* 30 (6), 695-716 (1998)
13. P. S. Addison, *Fractals and Chaos: An Illustrated Course*, Institute of Physics Publishing, Bristol, 1997.
14. H. E. Hurst, 'Long Term Storage Capacity of Reservoirs', *Trans. ASCE* 116, 770-808, (1951).
15. P. S. Addison and A. S. Ndumu, 'Engineering Applications of Fractional Brownian Motion: Self-Similar and Self-Affine Random Processes', *Fractals:* 7(2), 151-157. (1999).
16. Z. -M. Yin, 'New Methods for Simulation of Fractional Brownian Motion', *Jour. of Comp. Physics*, 127, 66-72 (1996).